

On the instability of a nonlocal conservation law

Afaf Bouharguane *

February 8, 2011

Abstract

We are interested in a nonlocal conservation law which describes the morphodynamics of sand dunes sheared by a fluid flow, recently proposed by Andrew C. Fowler and studied by [1, 2]. We prove that constant solutions of Fowler's equation are non-linearly unstable. We also illustrate this fact using a finite difference scheme.

Keywords: Fractional anti-diffusive equation, instability, finite difference schemes.

Mathematics Subject Classification: 35L65, 45K05, 35G25, 35C07, 35B35, 65M06.

1 Introduction

Partial Differential Equations with nonlocal or fractional operators are widely used to model scientific problems in mechanics, physics, signal processing and other subjects, see for example [3] and references therein. We consider in this paper a nonlocal conservation law which appears in the formation and dynamics of sand structures such as dunes and ripples [7, 11]. Namely, Fowler ([7, 8, 9]) introduced the following equation

$$\begin{cases} \partial_t u(t, x) + \partial_x \left(\frac{u^2}{2} \right) (t, x) + \mathcal{I}[u(t, \cdot)](x) - \partial_{xx}^2 u(t, x) = 0 & t \in (0, T), x \in \mathbb{R}, \\ u(0, x) = u_0(x) & x \in \mathbb{R}, \end{cases} \quad (1)$$

where $u = u(t, x)$ represents the dune height, $u_0 \in L^2(\mathbb{R})$ is an initial condition, T is any given positive time and \mathcal{I} is a nonlocal operator defined as follows: for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ and any $x \in \mathbb{R}$,

$$\mathcal{I}[\varphi](x) := \int_0^{+\infty} |\zeta|^{-\frac{1}{3}} \varphi''(x - \zeta) d\zeta. \quad (2)$$

Equation (1) is valid for a river flow over an erodible bottom $u(t, x)$ with slow variation. The nonlocal term appears after a subtle modeling of the basal shear stress. This operator appears also in the work of Kouakou & Lagrée [10, 11].

The nonlocal term \mathcal{I} can be seen as a fractional power of order $2/3$ of the Laplacian with the bad sign. Indeed, it has been proved [1]

$$\mathcal{F}(\mathcal{I}[\varphi] - \varphi'')(\xi) = \psi_{\mathcal{I}}(\xi) \mathcal{F}\varphi(\xi) \quad (3)$$

*Institut de Mathématiques et Modélisation de Montpellier, UMR 5149 CNRS, Université Montpellier 2, Place Eugène Bataillon, CC 051 34095 Montpellier, France. Email: bouharg@math.univ-montp2.fr

where

$$\psi_{\mathcal{I}}(\xi) = 4\pi^2 \xi^2 - a_{\mathcal{I}} |\xi|^{\frac{4}{3}} + i b_{\mathcal{I}} \xi |\xi|^{\frac{1}{3}}, \quad (4)$$

with $a_{\mathcal{I}}, b_{\mathcal{I}}$ positive constants and \mathcal{F} denotes the Fourier transform. One simple way to establish this fact is the derivation of a new formula for the operator \mathcal{I} [1]

$$\mathcal{I}[\varphi](x) = C_{\mathcal{I}} \int_{-\infty}^0 \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^{7/3}} dz, \quad (5)$$

with $C_{\mathcal{I}} = \frac{4}{9}$.

The operator $\mathcal{I}[u]$ is a weighted mean of second derivatives of u with the bad sign and has therefore an anti-diffusive effect and creates instabilities which are controlled by the diffusive operator $-\partial_{xx}^2$. This remarkable feature enabled to apply this model for signal processing. Indeed, the diffusion is used to reduce the noise whereas the nonlocal anti-diffusion is used to enhance the contrast [3].

Remark 1. For causal functions (i.e. $\varphi(x) = 0$ for $x < 0$), this operator is up to multiplicative constant, the Riemann-Liouville fractional derivative operator which is defined as follows [12]

$$\frac{1}{\Gamma(2/3)} \int_0^{+\infty} \frac{\varphi''(x-\xi)}{|\xi|^{1/3}} d\xi = \frac{d^{-2/3}}{dx^{-2/3}} \varphi''(x) = \frac{d^{4/3}}{dx^{4/3}} \varphi(x), \quad (6)$$

where Γ denotes the Euler function.

Recently, some results regarding this equation have been obtained, namely, existence of travelling-waves $u_{\phi}(t, x) = \phi(x - ct)$ where $\phi \in C_b^1(\mathbb{R})$ and $c \in \mathbb{R}$ represents wave velocity, the global well-posedness for L^2 -initial data, the failure of the maximum principle, the local-in-time well-posedness in a subspace of C_b^1 [1, 2] and the global well-posedness in a L^2 -neighbourhood of C_b^1 , solving namely for $u = u_{\phi} + v$, with $v \in L^2(\mathbb{R})$ [4]. For this purpose, the following Cauchy problem solved by the perturbation v has been considered:

$$\begin{cases} \partial_t v(t, x) + \partial_x \left(\frac{v^2}{2} + u_{\phi} v \right)(t, x) + \mathcal{I}[v(t, \cdot)](x) - \partial_{xx}^2 v(t, x) = 0 & t \in (0, T), x \in \mathbb{R}, \\ v(0, x) = v_0(x) & x \in \mathbb{R}, \end{cases} \quad (7)$$

where $v_0 \in L^2(\mathbb{R})$ is an initial perturbation and T is any given positive time.

Let us note also that any constant is solution of the Fowler equation. We shall prove below that these solutions are unstable.

To prove that a solution u_{ϕ} is unstable when ϕ is constant, we introduce the notion of *mild solution* (see Definition 1) based on Duhamel's formula (8). We also give some numerical results that illustrate this fact.

The remaining of this paper is organized as follows: in the next section, we define the notion of mild solution and we give some results. Section 3 contains the proof of the instability. We introduce in section 4 an explicit finite difference scheme for which we give numerical simulations to illustrate the theory of the previous section.

2 Duhamel formula and some results

Definition 1. Let u_ϕ be a constant, $T > 0$ and $v_0 \in L^2(\mathbb{R})$. We say that $v \in L^\infty((0, T); L^2(\mathbb{R}))$ is a **mild solution** to (7) if for any $t \in (0, T)$,

$$v(t, x) = K(t, \cdot) * v_0(x) - \int_0^t \partial_x K(t-s, \cdot) * \left(\frac{v^2}{2} \right) (s, \cdot)(x) ds \quad (8)$$

where $K(t, x) = \mathcal{F}^{-1} \left(e^{-t\phi_{\mathcal{I}}(\cdot)} \right) (x)$ is the kernel of the operator $\mathcal{I} - \partial_{xx}^2 + u_\phi \partial_x$ and $\phi_{\mathcal{I}}(\xi) = 4\pi^2 \xi^2 - a_{\mathcal{I}} |\xi|^{\frac{4}{3}} + i b_{\mathcal{I}} \xi |\xi|^{\frac{1}{3}} + i 2\pi u_\phi \xi$.

The expression (8) is the Duhamel formula and is obtained using the spatial Fourier transform. The use of this formula allows to prove the local-in-time existence with the help of the contracting fixed point theorem. The global existence is obtained thanks to L^2 a priori estimate. We refer to [1, 4] for the proof.

Lemma 1. Let $t > 0$. Then, $\partial_x K(t, \cdot) \in L^2(\mathbb{R})$ and satisfies

$$\|\partial_x K(t, \cdot)\|_{L^2(\mathbb{R})} \leq C \left(t^{-3/4} + e^{\alpha t} \right), \quad (9)$$

where C is a positive constant independent of t .

Proof. By Plancherel formula, we have

$$\begin{aligned} \|\partial_x K(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= \|\mathcal{F}(\partial_x K(t, \cdot))\|_{L^2(\mathbb{R})}^2 = \|\xi \mapsto 2i\pi \xi e^{-t\phi_{\mathcal{I}}(\xi)}\|_{L^2(\mathbb{R})}^2, \\ &= 2 \int_0^{+\infty} 4\pi^2 \xi^2 e^{-2t(4\pi^2 \xi^2 - a_{\mathcal{I}} \xi^{4/3})} d\xi. \end{aligned}$$

Let $\xi_0 > 0$ such that for all $\xi > \xi_0$,

$$4\pi^2 \xi^2 - a_{\mathcal{I}} \xi^{4/3} \geq \xi^2.$$

Then, if we denote by $\alpha = -\min \operatorname{Re}(\phi_{\mathcal{I}})$ (see Figure 2), we have

$$\begin{aligned} \|\partial_x K(t, \cdot)\|_{L^2(\mathbb{R})}^2 &\leq 8\pi^2 \int_0^{\xi_0} \xi^2 e^{-2t(4\pi^2 \xi^2 - a_{\mathcal{I}} \xi^{4/3})} d\xi + 8\pi^2 \int_{\xi_0}^{+\infty} \xi^2 e^{-2t(4\pi^2 \xi^2 - a_{\mathcal{I}} \xi^{4/3})} d\xi, \\ &\leq 8\pi^2 \int_0^{\xi_0} \xi^2 e^{2t\alpha} d\xi + 8\pi^2 \int_0^{+\infty} \xi^2 e^{-2t\xi^2} d\xi, \\ &\leq 8\pi^2 \frac{\xi_0^3}{3} e^{2t\alpha} + \frac{4\pi^2}{\sqrt{2}} t^{-3/2} \int_0^{+\infty} \xi^2 e^{-\xi^2} d\xi, \\ &\leq C \left(t^{-3/2} + e^{2\alpha t} \right), \end{aligned}$$

which completes the proof of this lemma. ■

Remark 2. Using again Plancherel formula, we have for any initial data v_0 with values in $L^2(\mathbb{R})$, the following L^2 -estimate [1]:

$$\|v(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{\alpha t} \|v_0\|_{L^2(\mathbb{R})}.$$

3 Instability

In this section, we give the proof of the instability.

Let us first note that a fundamental property of the kernel K is the non-positivity [1], see Figure 1. This feature enabled to prove the failure of the maximum principle. We use again this property to show that the constant solutions of the Fowler equation are unstable.

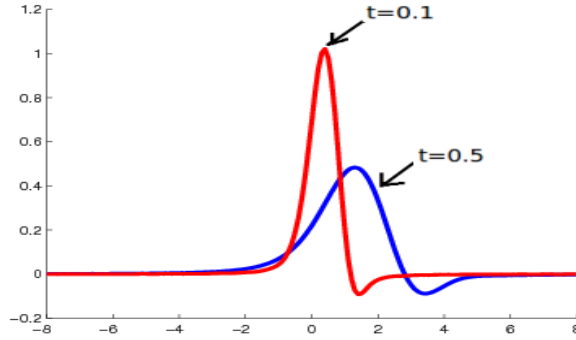


Figure 1: Evolution of the kernel K for $t = 0.1$ and $t = 0.5$ s

Theorem 1. Any constant solution u_ϕ of the Fowler equation is unstable, i.e. $\exists \varepsilon > 0, \forall \delta > 0, \exists v_0 \in L^2(\mathbb{R})$ an initial data with $\|v_0\|_{L^2(\mathbb{R})} \leq \delta$ and $t_0 > 0$ such that $\|v(t_0, \cdot)\|_{L^2(\mathbb{R})} > \varepsilon$, where $v(t, \cdot)$ is the solution of (7) with $v(0, \cdot) = v_0$.

Proof. We adapt the strategy used by De Bouard in [6], making it more elementary in our setting.

We denote by $S(t)$ the linear semigroup associated with equation (7) i.e. $S(t)w = K(t, \cdot) * w$. Hence, we have

$$T(t)v_0 := v(t, \cdot) = S(t)v_0 - \frac{1}{2} \int_0^t \partial_x K(t-s, \cdot) * v^2(s, \cdot) ds, \quad (10)$$

where T denotes the nonlinear semigroup associated with equation (7).

First, by Young inequality, Lemma 1 and Remark 2, we have

$$\begin{aligned} \|T(t)v_0 - S(t)v_0\|_{L^2(\mathbb{R})} &\leq \frac{1}{2} \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} \|v(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds, \\ &\leq \frac{C}{2} \int_0^t \left[(t-s)^{-3/4} + e^{\alpha(t-s)} \right] e^{2\alpha s} \|v_0\|_{L^2(\mathbb{R})}^2 ds, \\ &\leq b(t) \|v_0\|_{L^2(\mathbb{R})}^2, \end{aligned} \quad (11)$$

where $b(t) = \frac{C}{2} e^{2\alpha t} (4t^{1/4} + \frac{1}{\alpha} e^{\alpha t}) > 0$ for all $t \in (0, T)$.

Hence, we obtain for $t_0 > 0$ fixed,

$$\|T(t_0)v_0 - S(t_0)v_0\| \leq b_0 \|v_0\|_{L^2(\mathbb{R})}^2, \quad (12)$$

where $b_0 = b(t_0)$.

Assume now that the constant u_ϕ is a stable solution of the Fowler equation i.e.,

$$\forall \varepsilon > 0, \exists \eta > 0, \|v_0\| < \eta \Rightarrow \|v(t, \cdot)\|_{L^2(\mathbb{R})} < \varepsilon,$$

for all positive t . Let then

$$0 < \varepsilon < \frac{e^{\alpha t_0} - 1}{16b_0},$$

and let $\eta > 0$ be such that $\|v_0\|_{L^2(\mathbb{R})} < \eta \Rightarrow \|v(t, \cdot)\| \leq \varepsilon$, for all positive t .

Let N be an integer large enough such that

$$e^{\alpha t_0 N} \geq \frac{e^{\alpha t_0} - 1}{4\eta b_0}. \quad (13)$$

Let us now consider the following initial data: $v_0 = \delta w_0$, where $\mathcal{F}(w_0) = \frac{1}{\sqrt{d-c}} 1_{[c,d]}$, with $0 < c < d$ satisfying (see Figure 2):

- $\operatorname{Re}(\phi_{\mathcal{I}})(c) < \operatorname{Re}(\phi_{\mathcal{I}})(d) = -\beta < 0$,
- for all $\xi \in [c, d]$, $\operatorname{Re}(\phi_{\mathcal{I}})(\xi) \leq -\beta$,
- $\beta = \alpha - \gamma$ where $0 < \gamma < \frac{\ln(2)}{Nt_0}$.

And δ is defined as

$$\delta = e^{-\alpha N t_0} \frac{e^{\alpha t_0} - 1}{4b_0}. \quad (14)$$

The shape of w_0 is given by

$$w_0(x) = \frac{1}{\sqrt{d-c}} \begin{cases} 2 \frac{\sin(\frac{d-c}{2}x)}{x} e^{-i\frac{d+c}{2}x} & \text{if } x \neq 0, \\ d-c & \text{otherwise.} \end{cases}$$

It follows from (13) and (14) that $\|v_0\|_{L^2(\mathbb{R})} = \delta \leq \eta$, and thus for this initial data, we get that

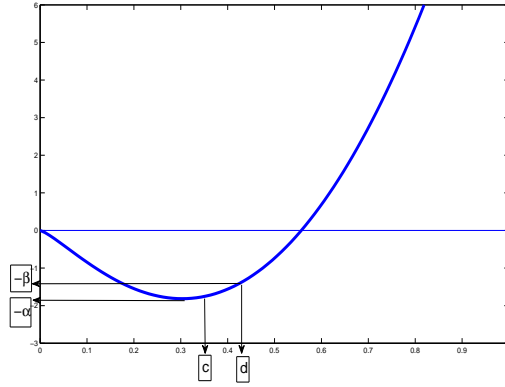


Figure 2: Behaviour of $\operatorname{Re}(\phi_{\mathcal{I}})$

$\|v(t, \cdot)\|_{L^2(\mathbb{R})} \leq \varepsilon$ for all $t > 0$.

For $n \in \{0, \dots, N\}$, we have

$$\begin{aligned} v(nt_0) &= S(nt_0)v_0 + \sum_{k=0}^{n-1} S((n-1-k)t_0) [v((k+1)t_0) - S(t_0)v(kt_0)], \\ &= S(nt_0)v_0 + \sum_{k=0}^{n-1} S((n-1-k)t_0) [T(t_0)v(kt_0) - S(t_0)v(kt_0)]. \end{aligned}$$

Hence, we get using (12) and the L^2 -estimate given in Remark 2

$$\begin{aligned}
\|v(nt_0) - S(nt_0)v_0\|_{L^2(\mathbb{R})} &\leq \sum_{k=0}^{n-1} e^{\alpha t_0(n-k-1)} \|v((k+1)t_0) - S(t_0)v(kt_0)\|_{L^2(\mathbb{R})}, \\
&\leq \sum_{k=0}^{n-1} e^{\alpha t_0(n-k-1)} b_0 \|v(kt_0)\|_{L^2(\mathbb{R})}^2, \\
&\leq \sum_{k=0}^{n-1} e^{\alpha t_0(n-k-1)} b_0 e^{2\alpha k t_0} \|v_0\|_{L^2(\mathbb{R})}^2, \\
&= \delta^2 b_0 e^{\alpha t_0(n-1)} \frac{e^{\alpha n t_0} - 1}{e^{\alpha t_0} - 1}, \\
&\leq \frac{\delta}{4} e^{\alpha t_0 n}.
\end{aligned}$$

Moreover, by Plancherel formula, we have

$$\begin{aligned}
\|S(t)v_0\|_{L^2(\mathbb{R})}^2 &= \|K(t, \cdot) * v_0\|_{L^2(\mathbb{R})}^2 = \|\mathcal{F}(K(t, \cdot) * v_0)\|_{L^2(\mathbb{R})}^2, \\
&= \int_{\mathbb{R}} |\mathcal{F}(K(t, \cdot))(\xi) \mathcal{F}(v_0)(\xi)|^2 d\xi, \\
&= \int_{\mathbb{R}} \frac{\delta^2}{d-c} e^{-2t \operatorname{Re}(\phi_{\mathcal{I}})(\xi)} d\xi, \\
&\geq e^{2\beta t} \|v_0\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

We finally infer that

$$\begin{aligned}
\|v(Nt_0)\|_{L^2(\mathbb{R})} &\geq \|S(Nt_0)v_0\|_{L^2(\mathbb{R})} - \|v(Nt_0) - S(Nt_0)v_0\|_{L^2(\mathbb{R})}, \\
&\geq \delta e^{\beta N t_0} - \frac{\delta}{4} e^{\alpha t_0 N}, \\
&\geq \frac{e^{\alpha t_0} - 1}{16b_0} > \varepsilon,
\end{aligned}$$

which gives us a contradiction and completes the proof of this theorem. ■

Remark 3. We can give a physical interpretation of this result: a flat profile $u_\phi = \text{constant}$ is unstable under the morphodynamics described by the Fowler model.

4 Numerical simulations

The spatial discretization is given by a set of points $x_j; j = 1, \dots, N$ and the discretization in time is represented by a sequence of times $t^0 = 0 < \dots < t^n < \dots < T$. For the sake of simplicity we will assume constant step size δx and δt in space and time, respectively. The discrete solution at a point will be represented by $u_j^n \approx u(t^n, x_j)$. The schemes consist in computing approximate values u_j^n of solution to (1) on $[n\delta t, (n+1)\delta t] \times [j\delta x, (j+1)\delta x]$ for $n \in \mathbb{N}$ and $j \in \mathbb{N}$ thanks to the following relation:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2} \frac{(u_j^n)^2 - (u_{j-1}^n)^2}{\delta x} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2} + \mathcal{I}_{\delta x}[u^n]_j = 0, \quad (15)$$

where $\mathcal{I}_{\delta x}$ is the discretization of the nonlocal term \mathcal{I} . We use a basic quadrature rule to approximate \mathcal{I} given by (2)

$$\mathcal{I}_{\delta x}[\varphi]_j = \delta x^{-4/3} \sum_{l=1}^{+\infty} l^{-1/3} (\varphi_{j-l+1} - 2\varphi_{j-l} + \varphi_{j-l-1}). \quad (16)$$

Let us note that for compactly supported initial datum the sum is a finite sum. In this section, we want to simulate the instability stated previously. To this aim we must be careful to distinguish between the instability which stems from the model and numerical instabilities, which could be caused by a careless discretization.

The numerical stability ensures that the difference between the approximate solution and the exact solution remains bounded for increasing t for $\delta x, \delta t$ given.

As noticed, the Fowler model amplifies slowly the low frequencies whereas the high frequencies are quickly dampened. Thus, the classical notion of A -stability (strong stability) and C -stability are not suitable nor desirable. A new definition of stability for this model has been considered in [5] and numerical stability criteria have been obtained.

In the following numerical test, we have to take care to choose δx and δt accurately following the stability condition, see [5]. We expose in Figure 3(a) the evolution of an initial flat bottom disturbed by a small bump for different times. As we can see the perturbation creates small ripples that grow with time without bound. This result proves that the perturbed solution goes away from the non-perturbed solution. This illustrates the instability of constant solutions for the Fowler model (1). Figure 3(b) shows the log-plot for the L^2 -norm evolution of this perturbed bottom, which confirms the validity of the estimate given in Remark 2.

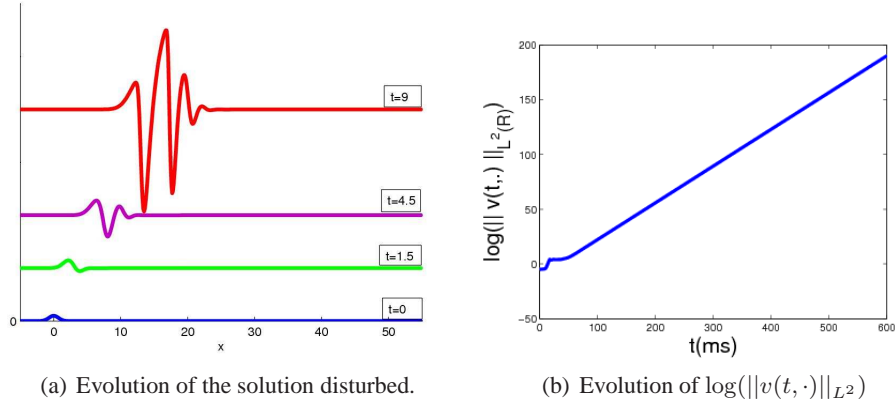


Figure 3: Evolution of a flat bottom disturbed by a small bump.

5 Acknowledgements

We thank Pascal Azerad and Clément Gallo for helpful comments. The author is supported by the ANR MATHOCEAN ANR-08-BLAN-0301-02

References

- [1] Alibaud N.; Azerad P.; D.Isebe "A non-monotone nonlocal conservation law for dune morphodynamics", *Differential and Integral Equations*, **23** (2010), pp. 155-188.
- [2] Alvarez-Samaniego B.; Azerad P. "Existence of travelling-wave and local well-posedness of the Fowler equation", *Disc. Cont. Dyn. Syst., Ser. B*, **12** (2009), pp. 671-692.
- [3] AZERAD P.; BOUHARGUANE A. ; CROUZET J.-F., Simultaneous denoising and enhancement of signals by a fractal conservation law, preprint: <http://arxiv4.library.cornell.edu/abs/1004.5193>.
- [4] BOUHARGUANE A. , *Global well-posedness and instability of a nonlocal conservation law* , in preparation.
- [5] AZERAD P.; BOUHARGUANE A. , *Finite difference schemes for a nonlocal conservation law*, in preparation.
- [6] DE BOUARD A., *Instability of stationary bubbles*, *SIAM J.Math Anal.*, **26** no. 3 (1995), pp. 566-582.
- [7] FOWLER A.C., *Dunes and drumlins*, *GEOMORPHOLOGICAL FLUID MECHANICS*, EDS. A. PROVENZALE AND N. BALMFORTH, SPRINGER-VERLAG, BERLIN, **211** (2001), pp. 430-454.
- [8] FOWLER A.C., *Evolution equations for dunes and drumlins*, *REV. R. ACAD. DE CIEN, SERIE A. MAT*, **96** (3) (2002), pp. 377-387.
- [9] FOWLER A.C, *Mathematics and the environment*, LECTURE NOTES [HTTP://WWW2.MATHS.OX.AC.UK/~FOWLER/COURSES/MATHENVO.HTML](http://www2.maths.ox.ac.uk/~fowler/courses/mathenvo.html).
- [10] KOUAKOU K.K.J.; LAGRÉE P-Y, *Evolution of a model dune in a shear flow*, *EUR. J. MECH. B FLUIDS*, **25** NO. 3 (2006), pp 348-359.
- [11] LAGRÉE P-Y; KOUAKOU K. , *Stability of an erodible bed in various shear flows*, *EUROPEAN PHYSICAL JOURNAL B - CONDENSED MATTER*, VOL. **47** (2005), pp 115-125.
- [12] PODLUBNY I., *An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.*, *MATHEMATICS IN SCIENCE AND ENGINEERING*, **198** ACADEMIC PRESS, SAN DIEGO, (1999).